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OSCILLATORY AND UNIMODAL PROPERTIES OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. The mathematical research presented here concerns a technique for studying certain properties of solutions to various types of differential equations.

SUMMARY

In previous papers, we have shown how to derive fundamental properties of the solutions of

$$u'' + p(x)u = 0$$

directly from the fact that it is the Euler equation associated with the functional

$$J(u) = \int_0^a [u'^2 - p(x)u^2]dx.$$

Here we wish to demonstrate a Sturm oscillation theorem and the unimodal property of the solution in analogous fashion, under the assumption that $p(x) > 0$.

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OSCILLATORY AND UNIMODAL PROPERTIES OF SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

1. INTRODUCTION

In two previous papers, we have shown how to derive fundamental properties of the solutions of

$$u'' + p(x)u = 0 \tag{1.1}$$

directly from the fact that it is the Euler equation associated with the functional

$$J(u) = \int_0^a [u'^2 - p(x)u^2]dx, \tag{1.2}$$

[1], [2]. Here we wish to demonstrate a Sturm oscillation theorem and the unimodal property of the solution in analogous fashion, under the assumption that $p(x) > 0$.

2. THE VARIATIONAL FORMULATION

Consider the problem of minimizing

$$K(u) = \frac{\int_0^a u'^2 dx}{\int_0^a p(x)u^2 dx} \tag{2.1}$$

over all functions $u(x)$ satisfying the conditions $u(0) = u(a) = 0$, and such that the numerator and denominator exist and are nonzero. This leads in the usual way to the equation

(2.2) $u'' + \lambda p(x)u = 0, \quad u(0) = u(a) = 0,$

with the minimum equal to the smallest characteristic value. If the equation

(2.3) $u'' + p(x)u = 0, \quad u(0) = u(a) = 0,$

with $p(x) > 0$, has a solution which is positive for $0 < x < a$, we know that $\lambda = 1$ is the smallest characteristic value and that $u(x)$ is, up to a normalization factor, the corresponding characteristic function. This result can be established without the use of Sturmian oscillation theorems, if we employ the Perron–Frobenius–Jentszch techniques of positive operators.

3. THE STURMIAN PROPERTY

Suppose that we are given the two equations

(3.1) $u'' + p(x)u = 0, \quad u(0) = 0,$

$v'' + q(x)v = 0, \quad v(0) = 0,$

with $q(x) \geq p(x) > 0$. We wish to demonstrate that the first zero of $v(x)$ cannot occur after the first zero of u.

We assume that the situations were the reverse, as indicated below (see Fig. 1). Then, from what has preceded,

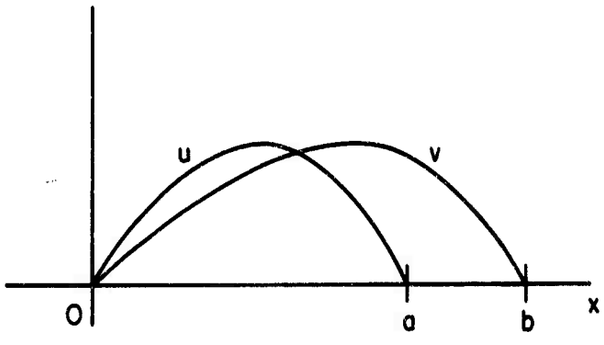

Fig. 1

$$(3.2)\qquad 1 = \min_{v} \frac{\int_0^b v'^2 dx}{\int_0^b q(x)v^2 dx} < \frac{\int_0^b w'^2 dx}{\int_0^b q(x)w^2 dx},$$

where $w(x)$ is another function defined over $[0,b]$. Choose $w(x)$ in the following way:

$$(3.3)\qquad w(x) = u(x), \quad 0 \le x \le a,$$

$$= 0, \qquad a \le x \le b.$$

Then (3.2) yields

$$(3.4)\qquad 1 < \frac{\int_0^a u'^2 dx}{\int_0^a q(x)u^2 dx} \le \frac{\int_0^a u'^2 dx}{\int_0^a p(x)u^2 dx} = 1,$$

a contradiction.

4. A UNIMODAL PROPERTY

Let us now demonstrate a familiar property of the first characteristic function $u(x)$, its unimodal character. Suppose that $u(x)$, the solution of

$$(4.1)\qquad u'' + p(x)u = 0, \quad u(0) = u(a) = 0,$$

with the property that $u(x) > 0$, $0 < x < a$, had the form in Fig. 2. Let us obtain from this function a function which yields a smaller value of $K(u)$, as defined in (2.1), in the fashion of Fig. 3. In the interval $[a_2,a_1]$, $v(x) = u(a_1)$, otherwise $v(x) = u(x)$.

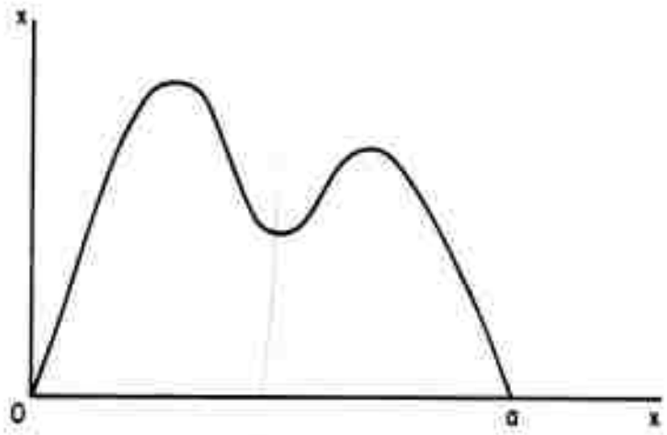

Fig.2

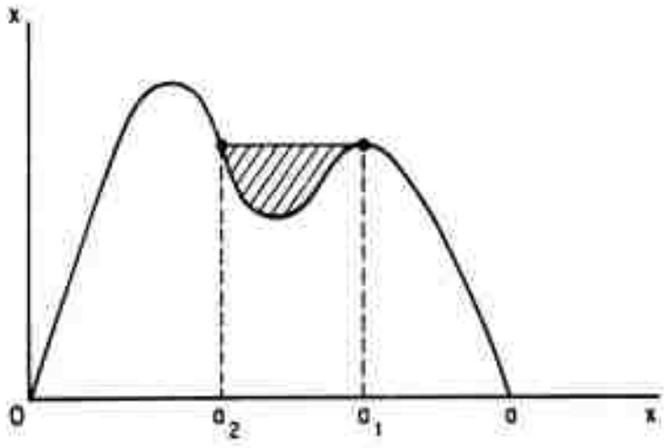

Fig.3

We see that

$$(4.2) \qquad \int_0^a v'^2 dx < \int_0^a u'^2 dx,$$

$$\int_0^a p(x)v^2 dx > \int_0^a p(x)u^2 dx.$$

Hence,

$$(4.3) \qquad K(v) < K(u),$$

a contradiction to the minimizing property of u.

5. DISCUSSION

One value of the foregoing technique is that it can be applied to study nonlinear ordinary and partial differential equations as well. Furthermore, we can study the properties of the higher characteristic functions by using relative minima.

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